Some exact solutions of reduced scalar Yukawa theory

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Abstract

The scalar Yukawa model, in which a complex scalar field, φ, interact via a real scalar field, χ, is reduced by using covariant Green functions. It is shown that exact few-particle eigenstates of the truncated QFT Hamiltonian can be obtained in the Feshbach-Villars formulation if an unorthodox "empty" vacuum state is used. Analytic solutions for the two-body case are obtained for massless chion exchange in 3+1 dimensions and for massive chion exchange in 1+1 dimensions. Comparison is made to ladder Bethe-Salpeter, Feynman-Schwinger and quasipotential results for massive chion exchange in 3+1. Equations for the three-body case are also obtained.

1. Introduction

The model scalar field theory, based on the Lagrangian density ($\hbar = c = 1$)

$$\begin{aligned}\mathcal{L} &= \partial^\nu \varphi^*(x)\,\partial_\nu \varphi(x) - m^2 \varphi^*(x)\varphi(x) \\ &\quad + \frac{1}{2}\partial^\nu \chi(x)\,\partial_\nu \chi(x) - \frac{1}{2}\mu^2 \chi^2(x) - g\varphi^*(x)\varphi(x)\,\chi(x)\end{aligned} \tag{1}$$

is often used as a prototype QFT in many studies. In particular, for the case $\mu = 0$, it is known as the Wick-Cutkosky model [1,2]. This model has been investigated in various formalisms, in addition to the original ladder Bethe-Salpeter formulation [3], including the light-cone formulation [4,5], and others (see refs. [6-9] and [18, 20-23] and citations therein).

We shall consider a reduced form of this theory in the present paper, in which the mediating chion field is partially eliminated by means of covariant Green functions, in a manner analogous to that discussed recently for QED [10,11]. In addition, we shall use the Feshbach-Villars (FV) formulation [12] for the complex phion field, and an unconventional "empty" vacuum state, as has been used recently to determine solutions of the $\lambda(\varphi^*\varphi)^2$ theory [13].

The fields φ and χ of the model (1) satisfy the equations

$$\partial^\nu \partial_\nu \chi(x) + \mu^2 \chi(x) = -g\varphi^*(x)\varphi(x), \tag{2}$$

$$\partial^\nu \partial_\nu \varphi(x) + m^2 \varphi(x) = -g\varphi(x)\chi(x), \tag{3}$$

and its conjugate.

Equation (2) has the formal solution

$$\chi(x) = \chi_0(x) + \int dx'\, D(x - x')\,\rho(x'), \tag{4}$$

where $dx = d^N x\, dt$ in $N+1$ dimensions, $\rho(x) = -g\varphi^*(x)\varphi(x)$, $\chi_0(x)$ satisfies the homogeneous (or free field) equation (eq. (2) with $g = 0$), while $D(x - x')$ is a covariant Green function (or chion propagator, in QFTheoretic language), such that

$$\left(\partial^\nu \partial_\nu + \mu^2\right) D(x - x') = \delta^{N+1}(x - x'). \tag{5}$$

Equation (5) does not specify $D(x - x')$ uniquely since, for example, any solution of the homogeneous equation can be added to it without invalidating (5). Boundary conditions based on physical considerations are used to pin down the form of D.

Substitution of the formal solution (4) into eq. (3) yields the equation

$$\partial^\nu \partial_\nu \varphi(x) + m^2 \varphi(x) = -g\varphi(x)\chi_0(x) - g\varphi(x) \int dx' D(x - x')\rho(x'). \tag{6}$$

Equation (6) is derivable from the action principle $\delta \int dx\, \mathcal{L} = 0$, corresponding to the Lagrangian density

$$\begin{aligned} \mathcal{L} = & \partial^\nu \varphi^*(x)\partial_\nu \varphi(x) - m^2 \varphi^*(x)\varphi(x) - g\varphi^*(x)\varphi(x)\, \chi_0(x) \\ & + \frac{1}{2} \int dx' \rho(x) D(x - x')\rho(x'), \end{aligned} \tag{7}$$

provided that $D(x - x') = D(x' - x)$.

The QFTs based on (1) and (7) are equivalent in that, in conventional covariant perturbation theory, they lead to the same invariant matrix elements in various order of perturbation theory. The difference is that, in the formulation based on (7), the interaction term that contains the propagator corresponds to Feynman diagrams involving virtual chions, while the term that contains χ_0 correspond to diagrams that cannot be generated using the term with $D(x - x')$, such as those with external (physical) chion lines.

We shall consider this scalar theory in the Feshbach-Villars (FV) formulation [12]. The reason for doing so is that this leads to a QFTheoretic Hamiltonian which is Schrödinger-like in form, for which exact eigensolutions can be readily written down. In the FV formulation, the field φ and its time-derivative $\dot{\varphi}$ are replaced by a two-component vector

$$\phi = \begin{bmatrix} u = \frac{1}{\sqrt{2m}}(m\varphi + i\dot{\varphi}) \\ v = \frac{1}{\sqrt{2m}}(m\varphi - i\dot{\varphi}) \end{bmatrix}, \tag{8}$$

so that, for example, $2m\, \varphi^* \varphi = (u^* + v^*)(u + v) = \phi^\dagger \eta\tau\phi$, where η and τ are the matrices

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{9}.$$

In the FV formulation the equation of motion (3) takes on the form

$$i\dot{\phi} = -\frac{1}{2m}\nabla^2 \tau\phi + m\eta\phi + \frac{g}{2m}\tau\phi\chi, \tag{10}$$

or, upon using (4), the form

$$\begin{aligned} i\dot{\phi} = & -\frac{1}{2m}\nabla^2 \tau\phi + m\eta\phi + \frac{g}{2m}\tau\phi\chi_0 \\ & + \frac{g}{2m}\tau\phi \int dx'\, D(x - x')\rho(x'), \end{aligned} \tag{11}$$

where $\rho = -g\varphi^*\varphi = -\frac{g}{2m}\phi^\dagger \eta\tau\phi$. Equation (11) is derivable from the Lagrangian density

$$\begin{aligned} \mathcal{L}_{FV}(x) = & \, i\phi^\dagger(x)\eta\dot{\phi}(x) - \frac{1}{2m}\nabla\overline{\phi}(x) \cdot \nabla\phi(x) - m\phi^\dagger(x)\phi(x) \\ & - \frac{g}{2m}\overline{\phi}(x)\phi(x)\chi_0(x) + \frac{1}{2}\int dx'\, \rho(x)D(x - x')\rho(x'), \end{aligned} \tag{12}$$

where $\overline{\phi} = \phi^\dagger \eta \tau$. Note that $\mathcal{L}$ of eq. (7) is not identical to $\mathcal{L}_{FV}$. Indeed $\mathcal{L} = \mathcal{L}_{FV} + \frac{\partial}{\partial t}(\varphi^* \dot{\varphi})$. However, they lead to identical equations of motion ((6) and (11)), and so are equivalent in this sense. Henceforth, we base our results on $\mathcal{L}_{FV}$.

We note that the momenta corresponding to u and v are

$$p_u = \frac{\partial \mathcal{L}}{\partial \dot{u}} = i u^*, \quad \text{and} \quad p_v = -i v^*,$$

that is, u^* and v^* are, in essence, the conjugate momenta, so that the Hamiltonian density is given by the expression

$$\mathcal{H}(x) = \phi^\dagger(x) \eta \hat{h}(x) \phi(x) + \frac{g}{2m} \overline{\phi}(x) \phi(x) \chi_0(x) - \frac{1}{2} \int dx' \, \rho(x) D(x - x') \rho(x'), \tag{13}$$

where $\hat{h}(x) = \tau(-\frac{1}{2m})\nabla^2 + m\,\eta$.

We use canonical equal-time quantization, whereupon the non-vanishing commutation relations are

$$[u(\mathbf{x}, t), p_u(\mathbf{y}, t)] = i\delta^N(\mathbf{x} - \mathbf{y}) \quad \text{and} \quad [v(\mathbf{x}, t), p_v(\mathbf{y}, t)] = i\delta^N(\mathbf{x} - \mathbf{y}), \tag{14}$$

or equivalently,

$$[\phi_a(\mathbf{x}, t), \phi_b^\dagger(\mathbf{y}, t)] = \eta_{ab} \delta^N(\mathbf{x} - \mathbf{y}), \qquad a, b = 1, 2 \tag{15}$$

and where $\phi^T = [\phi_1 = u, \phi_2 = v]$, while η_{ab} are elements of the η matrix (9). Using these commutation relations, the QFTheoretic Hamiltonian can be written as

$$H = \int d^N x \, [\mathcal{H}_0(x) + \mathcal{H}_\chi(x) + \mathcal{H}_I(x)], \tag{16}$$

where

$$\mathcal{H}_0(x) = \phi^\dagger(x) \eta \hat{h}(x) \phi(x), \tag{17}$$

$$\mathcal{H}_\chi(x) = \frac{g}{2m} \overline{\phi}(x) \phi(x) \chi_0(x), \tag{18}$$

and

$$\begin{aligned} \mathcal{H}_I(x) &= -\frac{g^2}{8m^2} \int dx' \, \overline{\phi}(x) \phi(x) D(x - x') \, \overline{\phi}(x') \phi(x') \\ &= -\frac{g^2}{8m^2} \int dx' \, D(x - x') \, \overline{\phi}(x) (\overline{\phi}(x') \phi(x')) \phi(x), \end{aligned} \tag{19}$$

and where we have used $\tau^2 = 0$ in the last step of (19). Note that no infinities are dropped upon normal ordering, since none arise on account of the $\tau^2 = 0$ property.

As already mentioned, $\mathcal{H}_I$ contains the covariant chion propagator, hence in conventional covariant perturbation theory it leads to Feynman diagrams with internal chion lines. On the other hand, $\mathcal{H}_\chi$ corresponds

to Feynman diagrams with external chions. However, we shall not pursue covariant perturbation theory in this work, and so shall not consider that approach further. Rather, we shall consider an approach that leads to some exact eigenstates of the Hamiltonian (16), but with $\mathcal{H}_\chi = 0$. Such a truncated Hamiltonian is appropriate for describing states for which there is no annihilation or decay involving the emission or absorbtion of real chions.

In the Schrödinger picture we can take $t = 0$. Therefore, we shall use the notation that, say $\phi(\mathbf{x}, t = 0) = \phi(\mathbf{x})$, etc., for QFT operators. This allows us to express (19) as

$$\mathcal{H}_I(\mathbf{x}) = -\frac{g^2}{8m^2} \int d^N x' \, G(\mathbf{x} - \mathbf{x}') \, \overline{\phi}(\mathbf{x}) (\overline{\phi}(\mathbf{x}') \phi(\mathbf{x}')) \phi(\mathbf{x}), \tag{20}$$

where

$$G(\mathbf{x} - \mathbf{x}') = \int_{-\infty}^{\infty} D(x - x') \, dt' = \frac{1}{(2\pi)^N} \int d^N p \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{\mathbf{p}^2 + \mu^2} \,. \tag{21}$$

Explicitly, for $N = 3$ this becomes

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi} \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}, \tag{22}$$

for $N = 2$ it is

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi} K_0(\mu|\mathbf{x} - \mathbf{x}'|), \tag{23}$$

where $K_0(z)$ is the modified Bessel function, whereas for $N = 1$ it has the form

$$G(x - x') = \frac{1}{2\mu} e^{-\mu|x-x'|} \,. \tag{24}$$

At this stage we proceed in an unorthodox fashion, and define an empty vacuum state, $|\tilde{0}\rangle$, such that

$$\phi_a |\tilde{0}\rangle = 0. \tag{25}$$

This is different from the conventional Dirac vacuum $|0\rangle$ (the "filled negative energy sea" vacuum), which is annihilated by only the positive frequency part of φ and by the negative frequency part of φ^* (see, for example, ref. [14], p. 38). With the definition (25), the state defined as

$$|\psi_1\rangle = \int d^N x \, \phi^\dagger(\mathbf{x}) \eta f(\mathbf{x}) |\tilde{0}\rangle, \tag{26}$$

where f($\mathbf{x}$) is a two-component vector, is an eigenstate of the truncated QFT Hamiltonian (16) ($\mathcal{H}_\chi = 0$) with eigenvalue E_1 provided that the $f(\mathbf{x})$ is a solution of the equation

$$\hat{h}(\mathbf{x}) f(\mathbf{x}) = E_1 f(\mathbf{x}). \tag{27}$$

This is just the free-particle Klein-Gordon equation for stationary states ($|\psi_1\rangle$ is insensitive to H_λ), in the FV formulation . It has, of course, all the usual negative-energy "pathologies" of the KG equation. The presence

of negative-energy solutions is a consequence of the use of vacuum (25). However, that is the price that has to be paid in order to obtain *exact* eigenstates of the truncated Hamiltonian (eq. (16) with $\mathcal{H}_\chi = 0$). We shall refer to $|\psi_1\rangle$ as a one-KG-particle state.

2. Two-particle eigenstates

We can define two-KG-particle states, analogously to (26):

$$|\psi_2\rangle = \int d^N x \, d^N y \; F_{a\,b}(\mathbf{x}, \mathbf{y}) \; \phi_a^\dagger(\mathbf{x}) \phi_b^\dagger(\mathbf{y}) |\tilde{0}\rangle, \tag{28}$$

where summation on repeated indices a and b is implied. This state is an eigenstate of the truncated QFT Hamiltonian ((16) with $\mathcal{H}_\chi = 0$) provided that the 2×2 coefficient matrix $F = [F_{a\,b}]$ is a solution of the two-body equation,

$$\eta \hat{h}(\mathbf{x}) \eta F(\mathbf{x}, \mathbf{y}) + [\eta \hat{h}(\mathbf{y}) \eta F^T(\mathbf{x}, \mathbf{y})]^T + V(\mathbf{x} - \mathbf{y}) \tau^T F(\mathbf{x}, \mathbf{y}) \tau = E_2 F(\mathbf{x}, \mathbf{y}), \tag{29}$$

where the superscript T stands for "transpose". The potential here is given by

$$V(\mathbf{x} - \mathbf{y}) = -\frac{g^2}{4m^2} G(\mathbf{x} - \mathbf{y}), \tag{30}$$

where G is specified in equations (21)-(24). Equation (29) is a relativistic two-body Klein-Gordon-Feshbach-Villars-like equation, with an attractive Yukawa interparticle interaction. If $V = 0$, then eq. (29) has the solution $F(\mathbf{x}, \mathbf{y}) = g_1(\mathbf{x}) g_2^T(\mathbf{y})$, where each $f_i(\mathbf{x}) = \eta \, g_i(\mathbf{x})$ is a solution of the free KG equation (27), with eigenenergy ε_i, where $E_2 = \varepsilon_1 + \varepsilon_2$, as would be expected.

In the rest frame, $\mathbf{P}_{\text{total}} |\psi_2\rangle = 0$, equation (29) simplifies to

$$\tilde{h}(\mathbf{r}) F(\mathbf{r}) + [\tilde{h}(\mathbf{r}) F^T(\mathbf{r})]^T + V(\mathbf{r}) \tau^T F(\mathbf{r}) \tau = E_2 F(\mathbf{r}), \tag{31}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$, $\tilde{h} = \eta \hat{h} \eta$, and $V(\mathbf{r}) = -\dfrac{g^2}{4m^2} G(\mathbf{r})$ in this case. It is useful to write out this equation in component form, with

$$F(\mathbf{r}) = \begin{bmatrix} s(\mathbf{r}) & t(\mathbf{r}) \\ u(\mathbf{r}) & v(\mathbf{r}) \end{bmatrix}, \tag{32}$$

namely

$$-\frac{1}{2m} \nabla^2 (2s - u - t) + V(s - t - u + v) + (2m - E_2) s = 0, \tag{33}$$

$$-\frac{1}{2m} \nabla^2 (s - v) + V(s - t - u + v) - E_2 t = 0, \tag{34}$$

$$-\frac{1}{2m} \nabla^2 (s - v) + V(s - t - u + v) - E_2 u = 0, \tag{35}$$

and

$$-\frac{1}{2m}\nabla^2(t+u-2v)+V(s-t-u+v)-(2m+E_2)v=0. \tag{36}$$

Equations (34) and (35) imply that $t(\mathbf{r})=u(\mathbf{r})$, so that only three equations survive:

$$(2m-E_2-\frac{1}{m}\nabla^2+V)s+(\frac{1}{m}\nabla^2-2V)t+Vv=0, \tag{37}$$

$$(-\frac{1}{2m}\nabla^2+V)s-(E_2+2V)t+(\frac{1}{2m}\nabla^2+V)v=0, \tag{38}$$

and

$$Vs+(-\frac{1}{m}\nabla^2-2V)t-(2m+E_2-\frac{1}{m}\nabla^2-V)v=0. \tag{39}$$

These equations have positive-energy solutions of the type $E_2=m+m+\cdots$, negative-energy solutions of the type $E_2=-m-m+\cdots$, and "mixed" type solutions with $E_2=m-m+\cdots$ (this is clear, for example, if $V=0$ and the particles are at rest).

For the positive-energy solutions, if we write $E_2=2m+\epsilon$, then in the non-relativistic limit $|(\epsilon+V+\frac{p^2}{m})v|\ll|mv|$ (and similarly for s and t), and so equation (38) and (39) show that t and v are small and doubly-small components respectively, by factors $O\left(\frac{\epsilon}{m}\right)$. Thereupon, equation (37) reduces to

$$-\frac{1}{m}\nabla^2 s(\mathbf{r})+V(\mathbf{r})s(\mathbf{r})=\epsilon s(\mathbf{r}), \tag{40}$$

which is the usual time-independent Schrödinger equation for the relative motion of two particles, each of mass m, interacting through the potential $V(\mathbf{r})$. Similarly, in the non-relativistic limit, v is the large component for the negative-energy solutions (i.e. $E_2=-(2m+\epsilon)$, and $s\to v$, $V\to -V$ in (40)), while t is the large component for the mixed energy solutions. This is obvious from the form of the free-particle solutions ($V=0$), which are

$$F(\mathbf{r})=s_0\begin{bmatrix}1 & (\frac{p}{\omega+m})^2\\ (\frac{p}{\omega+m})^2 & (\frac{\omega-m}{\omega+m})^2\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}} \underset{\frac{p}{m}\ll 1}{\longrightarrow} s_0\begin{bmatrix}1 & (\frac{p}{2m})^2\\ (\frac{p}{2m})^2 & (\frac{p}{2m})^4\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}}, \tag{41}$$

for $E_2=2\omega=2\sqrt{p^2+m^2}$,

$$F(\mathbf{r})=t_0\begin{bmatrix}\frac{p^2}{2m^2+p^2} & 1\\ 1 & \frac{p^2}{2m^2+p^2}\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}}, \tag{42}$$

for $E_2=0$, and

$$F(\mathbf{r})=v_0\begin{bmatrix}(\frac{\omega-m}{\omega+m})^2 & (\frac{p}{\omega+m})^2\\ (\frac{p}{\omega+m})^2 & 1\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}} \underset{\frac{p}{m}\ll 1}{\longrightarrow} v_0\begin{bmatrix}(\frac{p}{2m})^4 & (\frac{p}{2m})^2\\ (\frac{p}{2m})^2 & 1\end{bmatrix}e^{i\mathbf{p}\cdot\mathbf{r}}, \tag{43}$$

for $E_2=-2\omega=-2\sqrt{p^2+m^2}$, and where s_0, t_0 and v_0 are constants.

Equations (37)-(39) can be reduced by taking suitable linear combinations, whereupon it follows that ($E=E_2$)

$$(2m+E)v=(2m-E)s+2Et \tag{44}$$

and

$$[E(4m^2 - E^2) - 8m^2V]s = -[E(2m+E)^2 + 8m^2V]t. \tag{45}$$

It is easily verified that the free particle solutions (41) - (43), in particular, satisfy these relations. One can therefore write

$$s = [8m^2V + E(2m+E)^2]w \tag{45}$$

and

$$t = [8m^2V + E(E^2 - 4m^2)]w, \tag{46}$$

where w is a solution of ($E \neq 0$)

$$-\frac{1}{m}\nabla^2 w + \frac{m}{E/2}Vw = \frac{1}{m}((E/2)^2 - m^2)w. \tag{47}$$

Once (47) is solved for w, the components $s, t = u, v$ of the matrix F follow from (45) and (46).

Equation (47) is form-identical to the Schrödinger equation, and so can be solved in the same manner as the latter for both bound and continuum states. In general this has to be done numerically. In some cases, such as for $\mu = 0$ (massless chion exchange) in 3+1, and for $\mu \neq 0$ in 1+1, analytic solutions of Schrödinger's equation are known. We shall not discuss solution of equation (47) for the entire range of the parameters μ and g, for various N. Rather, we shall consider bound states for the two analytically solvable cases in some detail, leaving most of the rest for another time. We shall also examine bound states for one 3+1 case, with $\mu/m = 0.15$, numerically, since this case was studied in some detail recently by Nieuwenhuis and Tjon [8b].

3. Two-body bound states in 3+1 for massless chion exchange.

We consider, first, the solution of equation (47) for $N = 3$ and massless chion exchange (i.e. $\mu = 0$). In this case one can use the known hydrogenic solutions of the Schrödinger equation to obtain the solutions of eq. (47). Thus, for the bound states we obtain the result

$$(E/2)^2 - m^2 = -\frac{m^4\alpha^2}{n^2E^2}, \tag{48}$$

where $\alpha = \dfrac{g^2}{16\pi m^2}$, and $n = 1, 2, 3, \cdots$ is the principal quantum number. This yields the positive energy two-particle bound-state spectrum

$$E = m\sqrt{2\left(1 + \sqrt{1 - \left(\frac{\alpha}{n}\right)^2}\right)} = m\left(2 - \frac{1}{4}\left(\frac{\alpha}{n}\right)^2 - \frac{5}{64}\left(\frac{\alpha}{n}\right)^4 + \cdots\right) \quad , \tag{49}$$

which is seen to have the correct Rydberg non-relativistic limit. Note that the relativistic spectrum retains the "accidental" Coulomb degeneracy with respect to ℓ, unlike the Klein-Gordon spectrum for a static electromagnetic potential. This may seem surprising at first glance, but upon reflection it is not so. The potential $V = -eA^0 = -\dfrac{\alpha}{r}$ enters both linearly and quadratically for the KG equation in an external electromagnetic field $A^\mu = (A^0, \mathbf{0})$, whereas in the present scalar Yukawa theory the potential enters only linearly .

The two-particle bound-state wave functions corresponding to the eigenenergies (49) can be lifted similarly from the Schrödinger hydrogenic results. For example, the ground state wave function (unnormalized) corresponding to (49) with $n = 1$ is $w = e^{-\beta r}$, where $\beta = \dfrac{m^2\alpha}{E}$. The critical value of α beyond which E (eq. (49)) ceases to be real is, evidently, $\alpha = 1$ ($\alpha = n$, in general).

The question arises how these positive-energy bound-state solutions compare to corresponding results obtained in other formulations of this model. The original Wick-Cutkosky solutions [1,2] of the massless chion case in the ladder Bethe-Salpeter approximation, as well as the corresponding light-cone formulation [4,5], give the small-α expansion

$$E = m\left(2 - \frac{\alpha^2}{4n^2} - \frac{\alpha^3 \ln\alpha}{\pi n^2} + O(\alpha^3)\right). \tag{50}$$

This is different from the present results (49), for which the lowest-order correction to the Rydberg energy is $O(\alpha^4)$. The unusual $\alpha^3 \ln\alpha$ and α^3 terms (which have been termed a "disease" of the ladder Bethe-Salpeter Wick-Cutkosky solution [15]) are, apparently, an artifact of the *ladder* approximation. Variational-perturbative calculations of the scalar Yukawa model [9] that use a conventional (Dirac "filled negative-energy sea") vacuum, also yield no $O(\alpha^3 \ln\alpha, \alpha^3)$ terms. However the $O(\alpha^4)$ terms of [9] are different from those of equation (49). For example, for the ground state [9] give

$$E = m\left(2 - \frac{1}{4}\alpha^2 + \frac{19}{64}\alpha^4 + \cdots\right), \tag{51}$$

and there is no ℓ-degeneracy for states with $n > 1$. The disagreement between (51) and (49) at $O(\alpha^4)$ may be, in part, because virtual annihilation effects, which contribute at $O(\alpha^4)$, have not been included in the calculations of ref. [9] (i.e. eq. (51)). We might mention that a comparison of corresponding results for the Coulomb QED model (i.e. QED in the Coulomb gauge with the transverse $\alpha \cdot \mathbf{A}$ interaction turned off), namely calculations analogous to the present that use an "empty" vacuum [16], and conventional-vacuum variational-perturbative results [17] are in agreement at $O(\alpha^4)$. Thus, the reason for the lack of agreement between (49) and (51) beyond $O(\alpha^2)$ in the perturbative domain ($\alpha \ll 1$) is, at present, not clear to the author and is a matter that needs further investigation. The disagreement with the ladder Bethe-Salpeter

results (50) beyond $O(\alpha^2)$ is less surprising, given the oft-mentioned shortcomings of the *ladder* Bethe Salpeter approximation (see, for example, the discussion in refs. [8b], [15], [21], [24]).

It might be tempting to speculate that the reason for the disagreement in the perturbative domain lies buried in the use of the unconventional vacuum (25). However, as has already been mentioned, the use of such a vacuum in the Coulomb QED case, leads to no disagreement up to $O(\alpha^4)$. The present approach (with an empty vacuum) seems to correspond to the summation of a particular subset of diagrams in the conventional perturbative treatment (namely ladder and crossed-ladder diagrams; see also sect. 5). Such has been shown to be the case for the model theory where a (second) quantized Dirac field interacts with a classical (c-number) electromagnetic field [30]. However, since a detailed analysis along the lines of [30] has not been carried out for the scalar Yukawa model, this remains a point of speculation at this stage.

In the non-perturbative regime, the present results (which are not perturbative) decrease monotonically with increasing α to $E = \sqrt{2}\,m$ at $\dfrac{\alpha}{n} = 1$. This behaviour is characteristic of calculations of $E(\alpha)$ for this model in that $E(\alpha)$ decreases monotonically from $E(\alpha = 0) = 2m$ in all cases. However, the various approaches yield results that differ markedly in detail. For example, for the ground state ($n = 1$), the present results decrease much more rapidly with increasing α than any of the ladder Bethe-Salpeter calculations [2,4,5,6,8] or the Haag-expansion results of Raychaudhuri [18]. In particular, none of the latter give a restriction $0 \leq \alpha \leq 1$ as does the present analytic formula (49), and so no critical value of $E(\alpha = 1) = \sqrt{2}\,m$. Indeed the ladder Bethe-Salpeter and the variational -perturbative approximations all give values of E/m, for $n = 1$, which are near or above 1.9 at $\alpha = 1$, far above the present value of $1.414\cdots$, though the variational-perturbative results fall below the ladder Bethe-Salpeter ones for strong coupling. (A comparative plot of the ladder Bethe-Salpeter and variational- perturbative values is given in fig. 1 of ref. [9]). Raychaudhuri's Haag-expansion results for $\mu = 0$ are also very different in detail from the present analytic results (see table II of ref. [18]), however they do exhibit a critical value of α, but at a much larger value of $\alpha_c^{\rm Ray} \simeq 2.31$ (versus $\alpha_c = 1$ in our case) with the corresponding value of $E_c^{\rm Ray} = 1.30$ (vs. $1.414\cdots$ here).

There are no negative-energy bound state solutions in the present system since, as has been mentioned earlier, the potential effectively reverses sign for the negative-energy case (this also happens in the case of one-particle relativistic equations, such as the Klein-Gordon-Coulomb and Dirac-Coulomb equations). However, eq. (48) has "mixed-energy" type solutions with

$$E = m\sqrt{2\left(1 - \sqrt{1 - \left(\frac{\alpha}{n}\right)^2}\right)} = m\left(\frac{\alpha}{n} + \frac{1}{8}\left(\frac{\alpha}{n}\right)^3 + \frac{7}{128}\left(\frac{\alpha}{n}\right)^5 + \cdots\right) \quad . \tag{52}$$

These unphysical solutions do not have a Rydberg non-relativistic limit, and arise because of the retention of negative-energy solutions in the present formalism. For these mixed-energy solutions E increases mono-

tonically with increasing α from a value of $E = 0$ at $\alpha = 0$ to the value $E/m = \sqrt{2}$ at $\alpha = n$. It is of interest to note that the positive-energy and mixed-energy solutions join smoothly at $\alpha = n$. Thus, for $0 \leq \alpha < \alpha_c$, $E(\alpha)$ forms a continuous double-valued function, with the upper branch being the positive-energy solution and the lower branch being the mixed-energy solution. This is clear if one notes that eq. (48) can be recast into the equation of the semicircle $\left(\frac{E^2}{2m^2} - 1\right)^2 + \left(\frac{\alpha}{n}\right)^2 = 1$, $\alpha \geq 0$. Thus, eqs. (49) and (52) correspond, respectively, to the upper and lower branches of this semicircle. Solutions that include such "mixed-energy" behaviour arise in some other formulations of the relativistic two-body system. In particular Raychaudhuri's Haag-expansion results exhibit this "double-valued" behaviour (see fig. 5 of ref. [18]), although he does not identify the lower (in energy) branch as a "mixed-energy" phenomenon.

4. Two-body bound states in $1+1$.

Equation (47) in 1+1 corresponds to a one-dimensional Schrödinger equation with an exponential potential. This happens to be one of the not numerous cases for which the bound state eigenvalues can be expressed in terms of common analytic functions [19]. Thus, for the present case, the bound state eigenvalues are given by

$$J'_\nu(\gamma) = 0 \quad \text{even parity, and} \quad J_\nu(\gamma) = 0 \quad \text{odd parity,} \tag{53}$$

where $J_\nu(\gamma)$ is the usual Bessel function, while $\nu = 2\sqrt{m^2 - (E/2)^2}/\mu$ and $\gamma = g/\sqrt{\mu^3 E}$. We evaluated some solutions of equation (53) using the Maple programme. Their general behaviour is similar to the expressions (49) and (52), in that, for given m and μ there are positive-energy solutions, for which E/m starts from 2 when $g = 0$ and decreases monotonically with increasing g for $0 \leq g \leq g_c$, as well as mixed-energy solutions for which E/m start from 0 at $g = 0$ and increases monotonically with increasing g for $0 \leq g \leq g_c$. The two curves become coincident at $E(g_c)$, as was the case for the 3+1 solutions (49) and (52). A sample of the positive-energy ground-state solutions of (53), for $m/\mu = 6.944$, is given in table 1. We also list, for the ground state, corresponding non-relativistic results as well as perturbative and variational results obtained previously for the real scalar Yukawa model in 1+1 (i.e. real phions exchanging real chions) [19]. The results for E are listed in the table for various values of the parameter $\lambda = g/(4\sqrt{\pi}m)$. This parameter is not dimensionless in 1+1, unlike in 3+1 (where $\alpha = \lambda^2$ is dimensionless). The present relativistic results are seen to fall increasingly below the variational and perturbative results as λ increases. This is reminiscent of what was observed in 3+1. Also, the variational results (like the perturbative and non-relativistic ones) give no indication of a critical value of $\lambda_c = g_c/(4\sqrt{\pi}m)$ beyond which no real solution for E is obtained from (53).

The behaviour of the first few even-parity excited-state solutions of eq. (53) is qualitatively similar to that of the ground state. Table 2 is a list of the critical values of $E(\lambda_c)$ for the ground and first three even-parity excited states, for $m/\mu = 6.944$.

The mixed-energy solutions of eq. (53) for the ground state of the $m/\mu = 6.944$ case are listed in table 3. We have not done an investigation of the spectrum of energies for the entire range of values of m/μ in 1+1, as these can be readily obtained from eq. (53) as needed.

5. Two-body bound states in 3+1 for massive chion exchange.

Nieuwenhuis and Tjon recently reported an interesting study of the $\varphi^2\chi$ model using a Feynman - Schwinger formulation [8b] that contains all ladder and crossed ladder diagrams. They give a detailed comparison of their results with ladder Bethe-Salpeter and various "quasipotential" equations (i.e. modifications of the Bethe-Salpeter equation), among them the Blankenbecler-Sugar equation [20], the Gross equation [21] and the "equal-time equation" [22,23]. Since these various equations are written out and discussed in ref. [8b] we shall not repeat this here. Nieuwenhuis and Tjon find that the ladder Bethe-Salpeter results increasingly underestimate the two-body binding energy, as α increases, compared to their Feynman-Schwinger ladder-plus-crossed-ladder results, and, indeed, compared to the various quasipotential results (see fig. 1 of ref. [8b], in which E/m are plotted for $0 \leq \alpha < 0.93$). The various quasipotential results are distributed between the ladder Bethe-Salpeter and the Nieuwenhuis and Tjon Feynman-Schwinger results.

It is of interest, therefore, to compare the predictions of the present approach with those given in the work of Nieuwenhuis and Tjon [8b]. Thus, we calculated the positive-energy E/m values for various $0 \leq \alpha \leq \alpha_c$ for this $\mu/m = 0.15$ case. We used the Maple differential equation solver and the "shooting method" (see, for example, ref. [25] sect. 2.5). We tested the accuracy of this numerical procedure on the analytic cases discussed above in sections 3 and 4, and found that the analytic results could be reproduced with essentially arbitrary accuracy (we tried up to 10 figures for some cases). Since the results of the various approaches compared by Nieuwenhuis and Tjon , as well as ours, converge at small α (to the non-relativistic values), and since they all exhibit the same monotonically decreasing shape, we present here a list of E/m only at $\alpha = 0.9$ where the differences are quite marked. These results, in decreasing order of E/m, are: Ladder Bethe-Salpeter 1.963, Blankenbecler-Sugar 1.931, Gross 1.910, present results of eq. (47) 1.882, Gross (with retardation) 1.880, equal-time 1.860 and Nieuwenhuis and Tjon 1.770. These values, except for the present calculation, were read off Fig. 1 of Nieuwenhuis and Tjon [8b] and so the quoted accuracy of the last figure is doubtful. A more detailed list of our results for this $\mu/m = 0.15$ case is given in Table 4, alongside the

Nieuwenhuis and Tjon and Gross-equation results (taken from Fig.1 of [8b]).

A pictorial representation of most of these positive-energy solutions is given in Fig. 1. We plot our present results (solid line), along with the Ladder Bethe-Salpeter (dashed curve), the Blankenbecler-Sugar (diamonds), Gross with retardation (crosses), equal-time (stars) and Nieuwenhuis and Tjon (circles) values. All these results, save ours, are taken from Fig. 1 of Nieuwenhuis and Tjon [8b].

As can be seen, our results are most similar to those of Gross (with retardation). As such, they are substantially below the Ladder Bethe-Salpeter results, and close (or closer) to those with ladder and cross ladder diagrams. This suggests that the present formalism is equivalent, in some sense, to a summation of all ladder and crossed ladder diagrams in covariant perturbation theory. The fact that the present formalism is non-perturbative and contains the chion propagator as the effective potential, is supportive of this conjecture. However, it would be necessary to do an analysis like that of Guiasu and Koniuk [30] to demonstrate this explicitly and convincingly.

It is interesting to note that the Gross results exhibit the "double-valued" structure of $E(\alpha)$, with an upper "positive-energy" and a lower "mixed-energy" branch that join continuously at α_c^{G} (see inset in figure 1 of Nieuwenhuis and Tjon [8b]). This is precisely the behaviour that arises in the present formalism (e.g. eqs. [49] and [52] for $\mu = 0$). Indeed, the Gross critical value of $E(\alpha_c^{\mathrm{G}} = 1.28) \simeq 1.4m$ is quite close to the value that we obtain, namely $E(\alpha_c = 1.2087) = 1.48386m$. Our mixed-energy results for $\mu/m = 0.15$ are also similar to those obtained from the Gross eq., though our results lie somewhat higher. For example, we obtain $E(\alpha = 0.25) = 0.219, E(\alpha = 0.5) = 0.446, E(\alpha = 1.0) = 0.985$, whereas the Gross-eq. results are $E^{\mathrm{G}}(\alpha = 0.25) = 0.07, E^{\mathrm{G}}(\alpha = 0.5) = 0.36, E^{\mathrm{G}}(\alpha = 1.0) = 0.83$.

Table 5 contains a list of our $E(\alpha)$ for $\mu/m = 0.15$ in 3+1 for the two lowest excited states (labelled $n = 2$) with $\ell = 0$ (radial wave-functions $w(r)$ with one node) and $\ell = 1$ (nodeless $w(r)$). The general shape of $E(\alpha)$ for these excited states is similar to that of the ground state, except that the critical values of α (which are similar, but not coincident for these $n = 2$ states) are over twice as large as for the ground state. This is reminiscent of the $\alpha_c = n$ behaviour of the massless chion case ($\mu = 0$). There are mixed-energy branches of $E(\alpha)$ for the excited states, just as for the ground state (and the analytic $\mu = 0$ case), however we do not give a list of them here. Their behaviour is very similar to the case already discussed, in that $E(\alpha)$ increases monotonically from $E(\alpha = 0) = 0$ to $E(\alpha_c)$.

6. Three-body eigenstates

It is straightforward to write down three-body states analogous to the two-body state (28), namely

$$|\psi_3\rangle = \int d^N x_1\, d^N x_2\, d^N x_3\; F_{abc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, \phi_a^\dagger(\mathbf{x}_1)\phi_b^\dagger(\mathbf{x}_2)\phi_c^\dagger(\mathbf{x}_3)|\tilde{0}\rangle. \tag{54}$$

These are eigenstates of the Hamiltonian (16) (with $\mathcal{H}_\chi = 0$) corresponding to the eigenvalue E_3 provided that the $2^3 = 8$ coefficient functions $F_{abc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ are solutions of

$$\begin{aligned}
&\tilde{h}_{ak}(\mathbf{x}_1)F_{kbc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + \tilde{h}_{bk}(\mathbf{x}_2)F_{akc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + \tilde{h}_{ck}(\mathbf{x}_3)F_{abk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\
&\quad + V(\mathbf{x}_1 - \mathbf{x}_2)\tilde{\tau}_{ak_1}\tilde{\tau}_{bk_2}F_{k_1k_2c}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + V(\mathbf{x}_2 - \mathbf{x}_3)\tilde{\tau}_{bk_1}\tilde{\tau}_{ck_2}F_{ak_1k_2}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\
&\quad + V(\mathbf{x}_3 - \mathbf{x}_1)\tilde{\tau}_{ck_1}\tilde{\tau}_{ak_2}F_{k_1bk_2}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = E_3F_{abc}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\;.
\end{aligned} \tag{55}$$

Equations (55) are relativistic three-body Klein-Gordion-Feshbach-Villars-like equations. They are similar in structure to three-fermion equations derived previously [11, 26, 27], except that $\tilde{h}$ is then the Dirac operator and $a, b, c, ...$ are Dirac spinor indices, so that there are $4^3 = 64$ equations (not all of them independent, however). Solution of the equations (55) is much more challenging than for the two-body case, as it is for any relativistic three-body problem [28, 29, 31]. We shall not discuss the solution of these equations here. Generalizations for N-body eigenstates can be written down in an analogous fashion.

Concluding remarks

We have shown that the scalar Yukawa model can be recast in a form such that exact few-body eigenstates of the QFTheoretic Hamiltonian, in the canonic equal-time formalism, can be determined for the case where there are no free (physical) quanta of the mediating field (i.e. only virtual quanta). This is achieved by the partial elimination of the mediating field by means of Green functions, as well as by the use of the Feshbach-Villars formulation of scalar FT and the use of an "empty" vacuum state. This last requirement leads to the retention of negative-energy solutions, akin to the one-particle Klein-Gordon and Dirac equations.

We considered the solution of the resulting two-particle equations in some detail, for massless and massive quantum exchange, in 1+1 and 3+1 dimensions. Analytic solutions for the two-particle bound state eigenenergies were obtained for massive exchange in 1+1 (eq. 53), and for massless exchange in 3+1 (eq. 49). For the massive exchange case in 3+1 our results compare favourably with recent covariant Bethe-Salpeter based models, which include ladder and crossed-ladder diagrams (particularly those of the Gross

equation). However, our results give stronger two-body binding energies than results which contain only ladder diagrams (such as the ladder Bethe-Salpeter results), but weaker binding than results that go beyond ladder-plus-crossed-ladder effects (such as the Feynman- Schwinger calculations of Nieuwenhuis and Tjon).

The present approach can be used for relativistic three-body systems, and we derived such equations for the present scalar Yukawa model. It can also be used for other QFT models, such as spinor Yukawa model and QED.

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Table 1. Values of E_2 (ground state) for $m = 6.944$ (units: $\mu = 1$) in 1+1.

$\lambda = \frac{g}{4\sqrt{\pi}m}$	E_2 eq. (53)	E_2 non-rel.	E_2 [19] variational	E_2 [19] pert.
0.0	13.888	13.888	13.888	13.888
0.05	13.8866 8289	13.8866 8318	13.888	13.887
0.1	13.8746 2814	13.8746 5453	13.879	13.875
0.2	13.7885 3427	13.7898 3995	13.805	13.791
0.4	13.2754 9678	13.3205 9729	13.384	13.334
0.6	12.0275 4474	12.4161 0158	12.578	12.472
0.7	10.7258 6117	11.7945 8644	12.578	12.472
0.75	9.2844 6172	11.4408 8286	12.578	12.472
0.7589 10218*	8.2861 8251			
0.8		11.0583 6849	11.404	11.201
1.0		9.2380 4702	9.887	9.528
1.2		6.9492 3608	8.057	7.454
1.4		4.1878 1169	5.940	4.985
1.6		0.9507 0057	3.561	2.122
1.8		−2.7644 9139	0.942	

$^*\lambda_c$

Table 2. Values of $E_2(\lambda_c)$ in 1+1 for $m = 6.944$ (units: $\mu = 1$) for the ground state (labelled n=1) and first three even-parity excited states (labelled n=3,5,7).

n	λ_c	$E_2(\lambda_c)$	$E_2(\lambda_c)/m$
1	0.7589 1021 83	8.2862	1.1933
3	1.0429 3738 8	8.9174	1.2842
5	1.2762 9938 2	9.3428	1.3454
7	1.4983 1853 8	9.6932	1.4348

Table 3. Ground-state mixed-energy solutions $E_2(\lambda)$ in 1+1 for $m = 6.944$ (units: $\mu = 1$).

λ	$E_2(\lambda)$
0.1018 5940 83	0.1
0.1440 4069 17	0.2
0.2276 3468 26	0.5
0.3213 4930 72	1.0
0.4511 8881 61	2.0
0.6192 0329 79	4.0
0.7179 6575 40	6.0
0.7455 6999 26	7.0
0.7582 2397 98	8.0

Table 4. Values of E_2/m (ground state) for $\mu/m = 0.15$ in 3+1.

α	Niewenhuis and Tjon [8b]	Equal-time ref. [8b]	Gross eq. (with retard.)	Present results	Gross eq. ref. [8b]
0.3				1.999 536	
0.4	1.99			1.995 34	
0.5	1.98			1.986 30	
0.6	1.96	1.966	1.969	1.971 76	1.974
0.7	1.91	1.941	1.948	1.950 81	1.959
0.8	1.85	1.907	1.919	1.921 99	1.938
0.9	1.77	1.861	1.880	1.882 82	1.910
1.0				1.828 47	
1.1				1.746 64	
1.2				1.562 48	
1.205				1.534 20	
1.208				1.503 66	
1.2085				1.492 83	
1.2087*				1.483 86	

$^*\alpha_c$

Table 5. Values of E_2/m in 3+1 for $\mu/m = 0.15$ for the first two excited states.

α	E_2/m $(n = 2, \ell = 0)$ (one node)	α	E_2/m $(n = 2, \ell = 1)$ (no nodes)
1.0	1.999 954	1.4	1.999 031
1.25	1.996 28	1.5	1.995 22
1.5	1.985 75	2.0	1.951 8
2.0	1.937 1	2.5	1.846 1
2.5	1.822 7	2.8	1.686 2
2.7	1.725 7	2.82	1.663 9
2.8	1.625 3	2.84	1.635 2
2.82	1.579 5	2.85	1.616 1
2.825	1.556 8	2.86	1.589 2
2.827*	1.534 8	2.866*	1.558 1

$^*\alpha_c$